5. Appendix (Proofs of Statements)

Proof [Theorem 11] We start by showing the fulfillment of conditions (2.3a)-(2.3b). Based on condition (3.9), it can be directly concluded that satisfying conditions (3.10a)-(3.10b) ensures compliance with conditions (2.3a)-(2.3b), where $P = \big[\mathbb{L}(x)^\dagger \mathbb{R}_{0,T}\mathcal{H}(x)\big]^{-1}$. Next, we demonstrate that condition (2.3c) is also satisfied. Utilizing the quadratic form of the R-CBC, we have

$$\begin{aligned}\mathcal{B}(x^+) &= \big(A\mathcal{R}(x) + g(x)u + w\big)^\top P\big(A\mathcal{R}(x) + g(x)u + w\big)\\ &\overset{(3.6)}{=} \big((\mathbb{X}_{1,T} - (\mathbb{G}_{0,T} + \mathbb{W}_{0,T}) + g(x)\mathbb{U}_{0,T})\mathbb{Q}(x)x + w\big)^\top P\\ &\quad\times \big((\mathbb{X}_{1,T} - (\mathbb{G}_{0,T} + \mathbb{W}_{0,T}) + g(x)\mathbb{U}_{0,T})\mathbb{Q}(x)x + w\big)\\ &= x^\top\big((\mathbb{X}_{1,T} - (\mathbb{G}_{0,T} + \mathbb{W}_{0,T}) + g(x)\mathbb{U}_{0,T})\mathbb{Q}(x)\big)^\top P\\ &\quad\times \big((\mathbb{X}_{1,T} - (\mathbb{G}_{0,T} + \mathbb{W}_{0,T}) + g(x)\mathbb{U}_{0,T})\mathbb{Q}(x)\big)x\\ &\quad + w^\top P w + 2\,\underbrace{x^\top\big((\mathbb{X}_{1,T} - (\mathbb{G}_{0,T} + \mathbb{W}_{0,T}) + g(x)\mathbb{U}_{0,T})\mathbb{Q}(x))\big)^\top\sqrt{P}}_{a}\,\underbrace{\sqrt{P}w}_{b}.\end{aligned}$$

According to the Cauchy-Schwarz inequality (Bhatia and Davis, 1995), *i.e.,* $ab \leq |a||b|$, for any $a^\top, b \in \mathbb{R}^n$, followed by employing Young's inequality (Young, 1912), *i.e.,* $|a||b| \leq \frac{\pi}{2}|a|^2 + \frac{1}{2\pi}|b|^2$, for any $\pi \in \mathbb{R}^+$, one has

$$\begin{aligned}\mathcal{B}(x^+) &\leq x^\top\big((\mathbb{X}_{1,T} - (\mathbb{G}_{0,T} + \mathbb{W}_{0,T}) + g(x)\mathbb{U}_{0,T})\mathbb{Q}(x)\big)^\top P\\ &\quad\times \big((\mathbb{X}_{1,T} - (\mathbb{G}_{0,T} + \mathbb{W}_{0,T}) + g(x)\mathbb{U}_{0,T})\mathbb{Q}(x)\big)x\\ &\quad + \pi x^\top\big((\mathbb{X}_{1,T} - (\mathbb{G}_{0,T} + \mathbb{W}_{0,T}) + g(x)\mathbb{U}_{0,T})\mathbb{Q}(x)\big)^\top P\\ &\quad\times \big((\mathbb{X}_{1,T} - (\mathbb{G}_{0,T} + \mathbb{W}_{0,T}) + g(x)\mathbb{U}_{0,T})\mathbb{Q}(x)\big)x\\ &\quad + \frac{1}{\pi}\|\sqrt{P}\|^2\|w\|^2 + \|\sqrt{P}\|^2\|w\|^2\\ &= (1+\pi)x^\top\big((\mathbb{X}_{1,T} - (\mathbb{G}_{0,T} + \mathbb{W}_{0,T}) + g(x)\mathbb{U}_{0,T})\mathbb{Q}(x)\big)^\top P\\ &\quad\times \big((\mathbb{X}_{1,T} - (\mathbb{G}_{0,T} + \mathbb{W}_{0,T}) + g(x)\mathbb{U}_{0,T})\mathbb{Q}(x)\big)x + (1+\frac{1}{\pi})\|\sqrt{P}\|^2\|w\|^2.\end{aligned} \tag{5.1}$$

Since according to (3.9), one has $\mathbb{R}_{0,T}\mathcal{H}(x)P = \mathbb{L}(x)$, we utilize (3.3) and set $\mathbb{Q}(x) = \mathcal{H}(x)P$, implying that $\mathbb{Q}(x)P^{-1} = \mathcal{H}(x)$. Since $P = P^\top, P \succ 0$, and accordingly $P^\top P^{-1} = \mathbb{I}_n$, one can continue the inequality (5.1) as

$$\begin{aligned}\mathcal{B}(x^+) &\leq (1+\pi)x^\top\,\overbrace{P^\top\big[P^{-1}}^{\mathbb{I}_n}((\mathbb{X}_{1,T} - (\mathbb{G}_{0,T} + \mathbb{W}_{0,T}) + g(x)\mathbb{U}_{0,T})\mathbb{Q}(x))^\top P\\ &\quad\times ((\mathbb{X}_{1,T} - (\mathbb{G}_{0,T} + \mathbb{W}_{0,T}) + g(x)\mathbb{U}_{0,T})\mathbb{Q}(x))\,\underbrace{P^{-1}\big]P}_{\mathbb{I}_n}\,x + (1+\frac{1}{\pi})\|\sqrt{P}\|^2\|w\|^2\\ &\leq (1+\pi)x^\top P^\top\big[((\mathbb{X}_{1,T} - (\mathbb{G}_{0,T} + \mathbb{W}_{0,T}) + g(x)\mathbb{U}_{0,T})\mathcal{H}(x))^\top P\\ &\quad\times ((\mathbb{X}_{1,T} - (\mathbb{G}_{0,T} + \mathbb{W}_{0,T}) + g(x)\mathbb{U}_{0,T})\mathcal{H}(x))\big]Px + (1+\frac{1}{\pi})\|\sqrt{P}\|^2\|w\|^2.\end{aligned}$$

The expression for $\mathcal{B}(x)$ can be reformulated as $\mathcal{B}(x) = x^\top \underbrace{P^\top P^{-1}}_{\mathbb{I}_n} Px$. Then if

$$(1+\pi)\left((\mathbb{X}_{1,T} - (\mathbb{G}_{0,T} + \mathbb{W}_{0,T}) + g(x)\mathbb{U}_{0,T})\mathcal{H}(x)\right)^\top P \\ \times \left((\mathbb{X}_{1,T} - (\mathbb{G}_{0,T} + \mathbb{W}_{0,T}) + g(x)\mathbb{U}_{0,T})\mathcal{H}(x)\right) \leq \lambda P^{-1}, \quad (5.2)$$

one can show that $\mathcal{B}(x^+) \leq \lambda \mathcal{B}(x) + (1+\frac{1}{\pi})\|\sqrt{P}\|^2\|w\|^2$. From (5.2) , one has

$$\lambda P^{-1} - (1+\pi)\left((\mathbb{X}_{1,T} - (\mathbb{G}_{0,T} + \mathbb{W}_{0,T}) + g(x)\mathbb{U}_{0,T})\mathcal{H}(x)\right)^\top P \\ \times \left((\mathbb{X}_{1,T} - (\mathbb{G}_{0,T} + \mathbb{W}_{0,T}) + g(x)\mathbb{U}_{0,T})\mathcal{H}(x)\right) \geq 0. \quad (5.3)$$

According to Schur complement

$$\text{condition (5.3)} \quad \Leftrightarrow \underbrace{\begin{bmatrix} (\frac{1}{1+\pi})P^{-1} & (\mathbb{X}_{1,T} - (\mathbb{G}_{0,T} + \mathbb{W}_{0,T}) + g(x)\mathbb{U}_{0,T})\mathcal{H}(x) \\ \star & \lambda P^{-1} \end{bmatrix} \geq 0.}_{\text{condition (3.10c)}}$$

Due to the satisfaction of condition (3.10c), one has

$$\mathcal{B}(x^+) \leq \lambda \mathcal{B}(x) + \rho\|w\|^2,$$

with $\rho = (1+\frac{1}{\pi})\|\sqrt{P}\|^2$. Since $\|w\|^2 \leq \delta$ according to (2.2), we have $c = (1+\frac{1}{\pi})\|\sqrt{P}\|^2\delta$, which concludes the proof. ■

Proof [Lemma 14] Since $\varpi_0(x)$ is an SOS polynomial, it follows that $\varpi_0^\top(x)\beta_0(x) \geq 0$ for all $x \in X_0 = \{x \in \mathbb{R}^n \mid \beta_0(x) \geq 0\}$. Since $\mathcal{B}(x) = x^\top \left[\mathbb{L}(x)^\dagger \mathbb{R}_{0,T}\mathcal{H}(x)\right]^{-1} x$, where $P = \left[\mathbb{L}(x)^\dagger \mathbb{R}_{0,T}\mathcal{H}(x)\right]^{-1} \succ 0$, is also a non-negative SOS polynomial, it can be concluded that if condition (3.12a) is satisfied, condition (3.10a) will also hold. The same reasoning applies to conditions (3.12b) and (3.10b). Next, we show that condition (3.10c) is also satisfied. Since $\varpi(x)$ is an SOS polynomial, it implies that $\varpi^\top(x)\beta(x) \geq 0$ for all $x \in X = \{x \in \mathbb{R}^n \mid \beta(x) \geq 0\}$. Given that condition (3.12c) is also an SOS polynomial, one has

$$\begin{bmatrix} (\frac{1}{1+\pi})P^{-1} & (\mathbb{X}_{1,T} - (\mathbb{G}_{0,T} + \mathbb{W}_{0,T}) + g(x)\mathbb{U}_{0,T})\mathcal{H}(x) \\ \star & \lambda P^{-1} \end{bmatrix} - \varpi^\top(x)\beta(x)\mathbb{I}_{2n} \geq 0.$$

Thus, satisfying (3.12c) ensures that condition (3.10c) is also met, completing the proof. ■