5. Appendix (Proofs of Statements)

Proof [Theorem 1] The proof consists of two main parts: (i) $\forall (x,\hat{x}) \in \mathscr{R}$ one has $\|x-\hat{x}\| \leq \epsilon$, and (ii) $\forall (x,\hat{x}) \in \mathscr{R}$ and $\forall \hat{u} \in \hat{U}, \exists u \in U,$ such that $\forall x' \in A\mathcal{M}(x) + Bu, \exists \hat{x}' \in \Pi(A\mathcal{M}(\hat{x}) + B\hat{u})$, fulfilling $(x', \hat{x}') \in \mathscr{R}$. The first part follows directly from condition (5a) and the definition of the relation $\mathscr{R}$ in (6a):

$$\alpha\|x-\hat{x}\|^2 \leq \boldsymbol{\mathcal{S}}(x,\hat{x}) \leq \max\big(\bar{\rho}\nu, \bar{\psi}\big) \quad \rightarrow \quad \|x-\hat{x}\| \leq \left(\frac{\max\big(\bar{\rho}\nu,\bar{\psi}\big)}{\alpha}\right)^{\frac{1}{2}} = \epsilon.$$

We now move on to demonstrate the second part. Since

$$\boldsymbol{\mathcal{S}}\big(A\mathcal{M}(x) + Bu, \Pi(A\mathcal{M}(\hat{x}) + B\hat{u})\big) \leq \gamma\boldsymbol{\mathcal{S}}(x,\hat{x}) + \rho\|\hat{u}\| + \psi \leq \max\{\bar{\gamma}\boldsymbol{\mathcal{S}}(x,\hat{x}), \bar{\rho}\nu, \bar{\psi}\},$$

with $\bar{\gamma}, \bar{\rho}$ and $\bar{\psi}$ as

$$\bar{\gamma} = 1-(1-\eta_1)(1-\gamma), \quad \bar{\rho} = \frac{(1+\eta_2)\eta_3}{(1-\gamma)\eta_1}\rho, \quad \bar{\psi} = \frac{(1+\eta_2)\eta_3}{(1-\gamma)(\eta_3-1)\eta_1\eta_2}\psi,$$

for any $\eta_1, \eta_2 \in (0,1)$ and $\eta_3 \in (1,2)$, one has $\boldsymbol{\mathcal{S}}(x',\hat{x}') \leq \max\big(\bar{\rho}\nu, \bar{\psi}\big)$ given that $\bar{\gamma} \in (0,1)$ and $\boldsymbol{\mathcal{S}}(x,\hat{x}) \leq \max\big(\bar{\rho}\nu,\bar{\psi}\big)$ according to (6a), signifying that $(x',\hat{x}') \in \mathscr{R}$, completing the proof. ∎

Proof [Theorem 2] Initially, let us define

$$\boldsymbol{\mathcal{G}_1}(x) := \boldsymbol{\mathcal{Y}_1}(x)\mathcal{P}, \quad \text{and} \quad \boldsymbol{\mathcal{G}_2}(\hat{x}) := \boldsymbol{\mathcal{Y}_2}(\hat{x})\mathcal{P}. \tag{18}$$

Since $\mathcal{P} = \Xi^{-1}$, we consequently get $\boldsymbol{\mathcal{G}_1}(x) := \boldsymbol{\mathcal{Y}_1}(x)\Xi^{-1}$ and $\boldsymbol{\mathcal{G}_2}(\hat{x}) := \boldsymbol{\mathcal{Y}_2}(\hat{x})\Xi^{-1}$. Thus, the satisfaction of (9a) and (9b) directly follows from the fulfillment of (15a) and (15b), respectively.

Now, we proceed with demonstrating the fulfillment of condition (5b) under the satisfaction of conditions (15a)-(15e). To do so, by considering $\boldsymbol{\mathcal{S}}(x,\hat{x}) = (x-\hat{x})^\top\mathcal{P}(x-\hat{x})$, one has

$$\begin{aligned}&\boldsymbol{\mathcal{S}}\big(A\mathcal{M}(x) + Bu, \Pi(A\mathcal{M}(\hat{x}) + B\hat{u})\big)\\ &= \big(\underbrace{A\mathcal{M}(x) + Bu - \Pi(A\mathcal{M}(\hat{x}) + B\hat{u})}_{\clubsuit}\big)^\top\mathcal{P}\big(A\mathcal{M}(x) + Bu - \Pi(A\mathcal{M}(\hat{x}) + B\hat{u})\big).\end{aligned}$$

We first aim to compute a closed-form data-based representation for "♣" as follows:

$$\begin{aligned}&A\mathcal{M}(x) + Bu - \Pi(A\mathcal{M}(\hat{x}) + B\hat{u})\\ &\overset{(12)}{=} A\mathcal{M}(x) + B\boldsymbol{\mathcal{I}}\boldsymbol{\mathcal{G}_1}(x)x - B\hat{\boldsymbol{\mathcal{I}}}\boldsymbol{\mathcal{G}_2}(\hat{x})\hat{x} + B\hat{u} - \Pi(A\mathcal{M}(\hat{x}) + B\hat{u}).\end{aligned}$$

Now, by incorporating the term $A\mathcal{M}(\hat{x})$ through *addition and subtraction*, we have

$$\begin{aligned}&A\mathcal{M}(x) + Bu - \Pi(A\mathcal{M}(\hat{x}) + B\hat{u})\\ &= A\mathcal{M}(x) + B\boldsymbol{\mathcal{I}}\boldsymbol{\mathcal{G}_1}(x)x - A\mathcal{M}(\hat{x}) - B\hat{\boldsymbol{\mathcal{I}}}\boldsymbol{\mathcal{G}_2}(\hat{x})\hat{x} + A\mathcal{M}(\hat{x}) + B\hat{u} - \Pi(A\mathcal{M}(\hat{x}) + B\hat{u}).\end{aligned}$$

Subsequently, according to (13) in Lemma 1, we have

$$\begin{aligned}&A\mathcal{M}(x) + Bu - \Pi(A\mathcal{M}(\hat{x}) + B\hat{u})\\ &\overset{(13)}{=} \boldsymbol{\mathcal{O}}^{+}\boldsymbol{\mathcal{G}_1}(x)x - \hat{\boldsymbol{\mathcal{O}}}^{+}\boldsymbol{\mathcal{G}_2}(\hat{x})\hat{x} + A\mathcal{M}(\hat{x}) + B\hat{u} - \Pi(A\mathcal{M}(\hat{x}) + B\hat{u})\\ &\overset{(18)}{=} \boldsymbol{\mathcal{O}}^{+}\boldsymbol{\mathcal{Y}_1}(x)\mathcal{P}x - \hat{\boldsymbol{\mathcal{O}}}^{+}\boldsymbol{\mathcal{Y}_2}(\hat{x})\mathcal{P}\hat{x} + A\mathcal{M}(\hat{x}) + B\hat{u} - \Pi(A\mathcal{M}(\hat{x}) + B\hat{u}).\end{aligned}$$

Now, according to conditions (15c) and (15d), one has

$$\begin{aligned}&A\mathcal{M}(x) + Bu - \Pi(A\mathcal{M}(\hat{x}) + B\hat{u})\\&= \Theta\mathcal{P}(x - \hat{x}) + A\mathcal{M}(\hat{x}) + B\hat{u} - \Pi(A\mathcal{M}(\hat{x}) + B\hat{u}).\end{aligned} \tag{19}$$

Then by having the closed-form data-based representation of "♣" in (19), we have

$$\begin{aligned}&\boldsymbol{\mathcal{S}}\big(A\mathcal{M}(x) + Bu, \Pi(A\mathcal{M}(\hat{x}) + B\hat{u})\big)\\&= \big(\Theta\mathcal{P}(x - \hat{x}) + A\mathcal{M}(\hat{x}) + B\hat{u} - \Pi(A\mathcal{M}(\hat{x}) + B\hat{u})\big)^\top \mathcal{P}\big(\Theta\mathcal{P}(x - \hat{x}) + A\mathcal{M}(\hat{x}) + B\hat{u}\\&\quad - \Pi(A\mathcal{M}(\hat{x}) + B\hat{u})\big)\\&= (x - \hat{x})^\top \mathcal{P}\Theta^\top \mathcal{P}\Theta\mathcal{P}(x - \hat{x}) + 2\,\overbrace{(x - \hat{x})^\top \mathcal{P}\Theta^\top \sqrt{\mathcal{P}}}^{a}\,\overbrace{\sqrt{\mathcal{P}}(A\mathcal{M}(\hat{x}) + B\hat{u} - \Pi(A\mathcal{M}(\hat{x}) + B\hat{u}))}^{b}\\&\quad + (A\mathcal{M}(\hat{x}) + B\hat{u} - \Pi(A\mathcal{M}(\hat{x}) + B\hat{u}))^\top \mathcal{P}(A\mathcal{M}(\hat{x}) + B\hat{u} - \Pi(A\mathcal{M}(\hat{x}) + B\hat{u})).\end{aligned} \tag{20}$$

According to the Cauchy-Schwarz inequality (Bhatia and Davis, 1995), *i.e.*, $ab \leq \|a\|\|b\|$, for any $a^\top, b \in \mathbb{R}^n$, followed by employing Young's inequality (Young, 1912), *i.e.*, $\|a\|\|b\| \leq \frac{\mu}{2}\|a\|^2 + \frac{1}{2\mu}\|b\|^2$, for any $\mu \in \mathbb{R}_{>0}$, one has

$$\begin{aligned}&2(x - \hat{x})^\top \mathcal{P}\Theta^\top \sqrt{\mathcal{P}}\sqrt{\mathcal{P}}(A\mathcal{M}(\hat{x}) + B\hat{u} - \Pi(A\mathcal{M}(\hat{x}) + B\hat{u}))\\&\leq \mu(x - \hat{x})^\top \mathcal{P}\Theta^\top \mathcal{P}\Theta\mathcal{P}(x - \hat{x}) + \tfrac{1}{\mu}(A\mathcal{M}(\hat{x}) + B\hat{u} - \Pi(A\mathcal{M}(\hat{x}) + B\hat{u}))^\top \mathcal{P}(A\mathcal{M}(\hat{x}) + B\hat{u}\\&\quad - \Pi(A\mathcal{M}(\hat{x}) + B\hat{u})).\end{aligned} \tag{21}$$

Likewise, using Cauchy-Schwarz inequality, one can conclude that

$$\begin{aligned}&(A\mathcal{M}(\hat{x}) + B\hat{u} - \Pi(A\mathcal{M}(\hat{x}) + B\hat{u}))^\top \mathcal{P}(A\mathcal{M}(\hat{x}) + B\hat{u} - \Pi(A\mathcal{M}(\hat{x}) + B\hat{u}))\\&\leq \|\sqrt{\mathcal{P}}\|^2 \|(A\mathcal{M}(\hat{x}) + B\hat{u} - \Pi(A\mathcal{M}(\hat{x}) + B\hat{u}))\|^2 \overset{(4)}{\leq} \|\sqrt{\mathcal{P}}\|^2 \delta^2.\end{aligned} \tag{22}$$

Hence, by applying the bounds (21) and (22) to (20), one has

$$\boldsymbol{\mathcal{S}}\big(A\mathcal{M}(x) + Bu, \Pi(A\mathcal{M}(\hat{x}) + B\hat{u})\big) \leq (1 + \mu)(x - \hat{x})^\top \mathcal{P}\Theta^\top \mathcal{P}\Theta\mathcal{P}(x - \hat{x}) + (1 + \tfrac{1}{\mu})\|\sqrt{\mathcal{P}}\|^2 \delta^2.$$

According to the Schur complement (Zhang, 2006), and considering condition (15e) with $\mathcal{P} = \Xi^{-1}$, it is well-defined that

$$\begin{bmatrix}(\frac{1}{1+\mu})\mathcal{P}^{-1} & \Theta\\ \star & \gamma\mathcal{P}^{-1}\end{bmatrix} \succeq 0 \Leftrightarrow \gamma\mathcal{P}^{-1} - (1 + \mu)\Theta^\top \mathcal{P}\Theta \succeq 0 \Leftrightarrow \gamma\underbrace{\mathcal{P}\mathcal{P}^{-1}}_{\mathbb{I}_n}\mathcal{P} - (1 + \mu)\mathcal{P}\Theta^\top \mathcal{P}\Theta\mathcal{P} \succeq 0.$$

Thus, it is clear that

$$(1 + \mu)(x - \hat{x})^\top \mathcal{P}\Theta^\top \mathcal{P}\Theta\mathcal{P}(x - \hat{x}) \leq \gamma(x - \hat{x})^\top \mathcal{P}(x - \hat{x}).$$

Hence, one can deduce that

$$\boldsymbol{\mathcal{S}}\big(A\mathcal{M}(x) + Bu, \Pi(A\mathcal{M}(\hat{x}) + B\hat{u})\big) \leq \gamma\underbrace{(x - \hat{x})^\top \mathcal{P}(x - \hat{x})}_{\boldsymbol{\mathcal{S}}(x,\hat{x})} + \underbrace{(1 + \tfrac{1}{\mu})\|\sqrt{\mathcal{P}}\|^2 \delta^2}_{\psi},$$

implying that condition (5b) is satisfied under the fulfillment of conditions (15a)-(15e), with $\rho \equiv 0$ and $\psi = (1 + \frac{1}{\mu})\|\sqrt{\mathcal{P}}\|^2\delta^2$. To complete the proof, we note that

$$\lambda_{\min}(\mathcal{P})\|x - \hat{x}\|^2 \leq \boldsymbol{\mathcal{S}}(x, \hat{x}) \leq \lambda_{\max}(\mathcal{P})\|x - \hat{x}\|^2,$$

and thus, condition (5a) is satisfied with $\alpha = \lambda_{\min}(\mathcal{P})$, thereby concluding the proof. ■